PERIODIC ELECTROSTATIC FOCUSING OF A RIBBON BEAM

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Obtaining a solution of the equations of a beam satisfying certain conditions at the emitter is, as is known, only part of the problem. Any such solution determines the flow in an unbounded region, whereas actual beams have finite dimensions. To realize the flow described by the solution obtained, we must consider the problem of a system of focusing electrodes capable of producing a beam of given configuration. The solution of this problem reduces to the problem of analytic continuation of the potential given at the beam boundary together with its normal derivative into the charge-free region, i.e., to the Cauchy problem for the Laplace equation. The problem was first formulated and solved in [1] in relation to a space-charge beam. In [2–4], the concepts of [1] were generalized to the case of plane curved trajectories. The mathematical bases of the method of electrostatic focusing were considered in [5] (problems of existence, uniqueness, and correctness). For a number of flows, a solution was obtained in terms of contour integrals which are very difficult to evaluate [6]. In [7], an analytic solution of the problem of the formation of arbitrary axially symmetric beams is given. Transition to the complex domain and transformation of the Laplace equation to hyperbolic form made it possible to give the solution in a form more convenient for obtaining final results. Only a few analytic solutions in elementary functions and closed form are known for the problem of focusing stationary flows [1, 4, 8–15] (plane diode [1, 13, 15], plane magnetron [4, 8, 9], hyperbolic [10] and elliptic [11, 12] beams, flow along circles and spirals in some nonuniform magnetic fields [14]). In [16] electrodes were determined for several nonstationary beams.

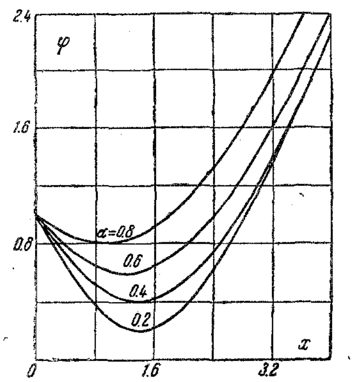

Fig. 1

Reference [17] is a study of the modes which may exist for monoenergetic nonrelativistic flows of particles of like charge between parallel planes. In [15] the problem of focusing a ribbon beam is solved for arbitrary conditions at the emitter and a monotonely changing potential. Below, we consider the case (case C in the terminology of [14]) when the potential between the electrodes has an extremum (a minimum for electrons) (§ 1). The exact solution is compared with the approximate one given in [18]. § 2 shows how the results of the preceding section can be used to obtain an exact analytic solution of the problem of periodically focusing a ribbon beam [19, 20]. The results of § 2 are compared with references [20, 21], in which an approximate solution of the problem is given.

§1. Let us consider the case when the potential has an extremum. We shall assume that φ_1 is the emitter potential at x = 0 and that the velocity there is equal to $(-2\eta\varphi_1)^{1/2}$. Going over to the dimensionless variables x^0, φ^0, measuring lengths along the x axis in units of a, and referring the potential to φ_1, we have

$$x = ax^\circ, \qquad \varphi = \varphi_1 \varphi^\circ, \qquad a = \left(\sqrt{2|\eta|}\,/\,9\pi\,|j_0|\right)^{1/2} \varphi_1^{3/4}.$$

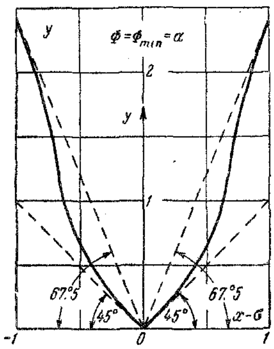

Fig. 2

We consider a to represent the distance at which (in accordance with the Child-Langmuir solution) the potential difference $\delta\varphi = \varphi_1$ induces a current of density j_0; η is the charge-to-mass ratio of the particle. In this case, the solution of the beam equations in dimensionless variables (the symbol of nondimensionality is omitted) is given by [17]

$$x = \mp(\varphi^{1/2} + 2\alpha^{1/2})\sqrt{\varphi^{1/2} - \alpha^{1/2}} + (1 + 2\alpha^{1/2})\sqrt{1 - \alpha^{1/2}}. \quad (1.1)$$

The minus sign in expression (1.1) holds for the interval $1 \geqslant \varphi \geqslant \alpha$, $0 \leqslant x \leqslant \sigma$; the plus sign holds for $\varphi \geqslant \alpha$, $x \geqslant \sigma$; here $\alpha = \varphi(\sigma) = \varphi_{min}$. The type-C potential distribution $\varphi = \varphi(x)$ is shown in Fig. 1 for various values of a.

We shall assume that the charges occupy the region $x \geqslant 0$, $y \leqslant 0$. In order to obtain the equations of the focusing electrodes, we replace x by $z = x + iy$ in (1.1)

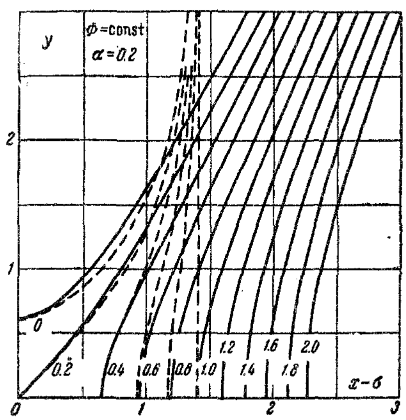

Fig. 3

and represent φ in the form $\varphi = \Phi + i\Psi$. Then, separating the real and imaginary parts, we arrive at an expression of the form

$$x = x(\Phi, \Psi; \alpha), \qquad y = y(\Phi, \Psi; \alpha).$$

Setting $\Phi = \Phi_0$, we obtain the parametric equation of the electrode with potential $\Phi = \Phi_0$:

$$x = x(\Phi_0, \Psi; \alpha), \qquad y = y(\Phi_0, \Psi; \alpha).$$

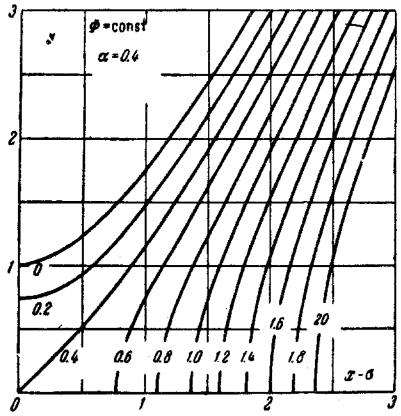

Fig. 4

For the flow described by (1.1), the equipotential surfaces in the region outside the beam are given by the equations

$$x-(1+2\alpha^{1/2})\sqrt{1-\alpha^{1/2}} = x-\sigma =$$
$$= \mp 2^{-1/2}\Big[\Big(\sqrt{1/2\,(r+\Phi)}+2\alpha^{1/2}\Big)\sqrt{R+\sqrt{1/2\,(r+\Phi)}-\alpha^{1/2}} -$$
$$-\sqrt{1/2\,(r-\Phi)}\sqrt{R-\sqrt{1/2\,(r+\Phi)}+\alpha^{1/2}}\Big]; \qquad (1.2)$$
$$y = 2^{-1/2}\Big[\Big(\sqrt{1/2\,(r+\Phi)}+2\alpha^{1/2}\Big)\sqrt{R-\sqrt{1/2\,(r+\Phi)}+\alpha^{1/2}} +$$
$$+\sqrt{1/2\,(r-\Phi)}\sqrt{R+\sqrt{1/2\,(r+\Phi)}-\alpha^{1/2}}\Big];$$
$$r=\sqrt{\Phi^2+\Psi^2}, \qquad R=\sqrt{r-\sqrt{2\alpha(r+\Phi)}+\alpha}\,.$$

For $\Psi \to \infty$ formulas (1.2) yield the following expressions for x and y:

$$x-\sigma = \mp 1/2\Big(\sqrt{1+1/2\sqrt{2}}-\sqrt{1-1/2\sqrt{2}}\Big)\Psi^{3/4};$$
$$y = 1/2\Big(\sqrt{1-1/2\sqrt{2}}+\sqrt{1+1/2\sqrt{2}}\Big)\Psi^{3/4}.$$

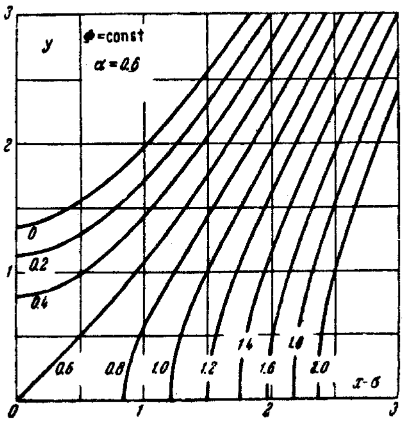

Fig. 5

Thus, the straight lines $y = \mp(1+\sqrt{2})(x-\sigma)$ approaching the beam boundary at an angle of 67.5° are asymptotes of the family of equipotential curves $\Phi = \text{const}$. Remembering that the zero value for the parameter Ψ corresponds to the beam boundary at $y = 0$, we see that all electrodes with potential $\Phi > \alpha$ approach it at right angles, since for small Ψ

$$x = \lambda + \mu\Psi^2, \qquad y = \nu\Psi.$$
$$(\lambda,\ \mu,\ \nu = \text{const})$$

For $\Phi = \alpha$ and small Ψ, Eqs. (1.2) yield*

$$x-\sigma = \mp 3/2\,\alpha^{1/4}\sqrt{\Psi};$$
$$y = 3/2\,\alpha^{1/4}\sqrt{\Psi}.$$

Consequently, the equipotential surfaces $\Phi = \alpha$ form an angle of 45° with the beam boundary. Figure 2 is a diagram of the electrode with minimum potential $\Phi = \alpha$.

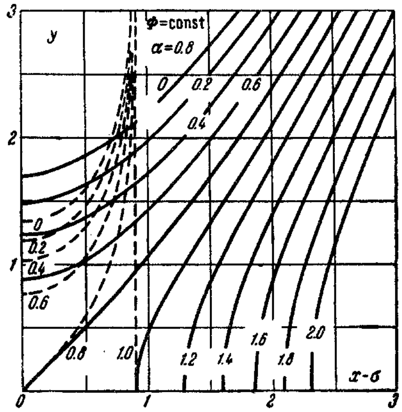

Fig. 6

Since the potential $\varphi(x)$ is symmetric with respect to the nonsingular point $x = \sigma$ in the interval $0 \leqslant x \leqslant 2\sigma$ for electrodes with potential $\Phi < \alpha$, we have

$$dy\,/\,dx = 0 \quad \text{for} \quad x = \sigma, \quad \Phi < \alpha.$$

When a virtual emitter ($\alpha = 0$) is created between the electrodes and the current is partially reflected [17], on both sides of $x = \sigma$, the electrodes coincide with those determined in [1]. The point $x = \sigma$ is a singular point ($d^n\varphi/dx^n|_{x=\sigma} = \infty$, $n = 2, 3, \ldots$) and, upon analytic continuation of the potential, generates the line $x = \sigma$ on which Φ and $\partial\Phi/\partial x$ experience a discontinuity.

Curves $\Phi = \text{const}$ (solid curves) are represented in Figs. 3–6 for various values of α. The equipotential surfaces $\alpha \leqslant \Phi \leqslant 1$ at $0 \leqslant x \leqslant \sigma$ are obtained from the reflection of the surfaces $\alpha \leqslant \Phi \leqslant 1$ at $x \geqslant \alpha$ about the y axis.

*For $u_0 \neq 0$, $\varepsilon_0 = 0$ (case 2 of [15]), x and y behave in exactly the same way at small Ψ; therefore the curve $\Phi = 0$ (Fig. 6) approaches the beam boundary at an angle of 45° and not 90°.

Thus, for nonzero velocity and field at the emitter u_0, $\mathcal{E}_0 \neq 0$ and emission limited by temperature $u_0 = 0$, $\mathcal{E}_0 \neq 0$ [13, 15], the zero equipotential approaches the beam boundary at right angles; for $u_0 \neq 0$, $\mathcal{E}_0 = 0$ (or minimum potential) at an angle of 45°; and for $u_0 = \mathcal{E}_0 = 0$ at an angle of 67.5° [1]. There is no continuous dependence of the slope of the zero equipotential on u_0, $\mathcal{E}_0$.

It can be shown, as in [5] for an angle of 67.5°, that 45° and 90° angles are characteristic not only of a plane diode but also of emission from an arbitrary surface. The discrete dependence of the slope ϑ_0 of the zero equipotential on u_0, $\mathcal{E}_0$ is retained even at relativistic velocities despite the statement in [22] that there is a continuous variation of ϑ_0 with change in collector potential.

Reference [18] gives an approximate solution of the problem in question for the case of a potential minimum in the middle of the interelectrode space. It is based on the approximation of the potential at the beam boundary by a parabola; for $\alpha > 0.71$ the error is no greater than 0.5%. The central electrode proved to be a plane approaching the beam boundary at an angle of 45°.

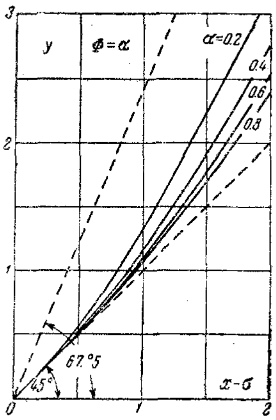

Fig. 7

Figure 7 gives curves $\Phi = \alpha$ in $x - \sigma$, y coordinates for different α. It is clear that the difference between the exact solution and the approximate one $y_a = x - \sigma$ quickly becomes noticeable. To compare these solutions for $\alpha = 0.8$, we present values of $= 1 - y_a/y$, where y_a is the approximate, and y the exact value of the electrode ordinate, computed for several values of $x - \sigma$:

$x - \sigma =$	0.142	0.283	0.316	0.446	0.545	1.07	1.65	2.11
δ, % =	0.175	0.377	0.585	1.04	1.57	5.67	12.4	17.3.

The central electrode $\Phi = 0.8$ is determined with the same accuracy as the potential at the beam boundary for $x - \sigma < 0.3$.

As is known, the Cauchy problem for the Laplace equation is incorrect: a small perturbation in the initial conditions at the boundary causes a change in the solution that increases without bound with distance from the boundary [5, 23, 24]. This instability makes it difficult to perform numerical integration and seek a solution by expanding the initial conditions in series or assigning them approximately [25–27]. Reference [25] deals with the case when the potential at the boundary $\varphi = (1 + x^2)^{-1}$ is approximated with an error not greater than 1.5% by a tenth-degree polynomial. A comparison of the approximate and exact solutions of the Cauchy problem reveals a very large difference between the two families of equipotentials; the exact solution has a singular point (0, 1) which, naturally, is not preserved in the indicated approximation. The branch point that appears in the approximate solution is absent in the exact solution. It is shown in [26] that the equipotential surfaces determined using three and six terms of the expansion differ greatly. In [28] a numerical method for solving the Pierce problem is proposed, in which the Laplace equation is written in finite differences. In [27] the calculations of [28] are repeated with a higher degree of accuracy and the solution is found to oscillate strongly: as the step approaches zero, the numerical solution approaches the exact one. In [29] the rate of error increase is estimated and calculation methods which must be stable are proposed. In [30] yet another reason, accounting, at least in part, for the results of [27], is suggested: higher derivatives are used in the solution in [28] whereas the value of the potential and its normal derivative at the boundary uniquely determines the solution in the entire region. The finite-difference method of integration may be applied successfully if the Laplace equation is reduced to hyperbolic type upon transition to the complex domain.

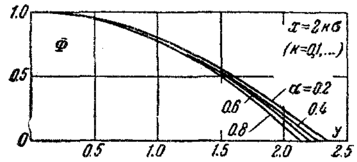

Fig. 8

The foregoing gives yet another example of the instability of the solution for a problem practical interest.

In a number of studies [26, 33–38] the potential is expanded near the beam boundary in power series. In view of the nonlinearity of the beam equations it is not possible to prove that these series converge absolutely. Therefore, to construct a solution in this region with a given accuracy is likewise not possible [5]. In obtaining solutions by this method, one should remember that they are close to the exact solution only in the immediate neighborhood of the beam boundary.

§2. The results of the preceding section can be used for the exact analytic solution of the problem of periodic focusing of a ribbon beam [19–21]. The potential distribution within the interval $2k\sigma \leq x \leq 2 \times (k + 1)\sigma$ ($k = 0, 1, \ldots$) is given by Eqs. (1.1), where $\varphi(0) = \varphi(2\sigma) = 1$ and $\varphi(\sigma) = \alpha$ (Fig. 1). The discontinuity in $d\varphi/dx$ at the ends of each interval requires the presence in the beam of grids at a potential equal to 1. A particular case of the problem of §1 is the determination of the electrodes for one of the elements of a periodic focusing system $2k\sigma \leqslant x \leqslant 2(k + 1)\sigma$. It remains to investigate the coupling of two such elements, since the solution in §1 is not periodic.

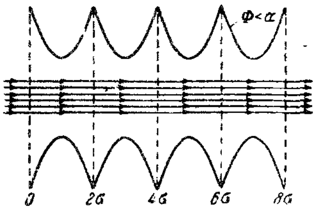

Fig. 9

A similar situation exists in the problem of focusing an arbitrary number of parallel ribbon beams. References [5, 39] show that a solution $\Phi = \Phi(x, y)$, continuous together with its first derivatives, may not exist in the region between two beams. In [5] this problem is presented as an example of nonsatisfaction of the uniqueness theorem for the case when the Cauchy conditions are given at a nonanalytic boundary. In [39] the proof is based on the fact that the analytic continuation of the potential given at the boundary of one beam does not necessarily coincide with the continuation of the potential from the boundary of the second beam; this leads to a physically inadmissible multivalued solution. The impossibility of a continuous solution also results from the following considerations. The potential $\Phi(x, y)$ will be continuous together with its derivatives if there exists a conformal mapping of the plane with one discarded pole onto the plane with a system of such cuts. The potential distribution along the beam boundaries must be invariant with respect to this mapping, since both planes are physical planes. In determining the focusing electrodes for plane curved trajectories [2, 4], this requirement was superfluous, since the u, v plane obtained upon mapping the beam boundaries onto the real axis v = 0 was an auxiliary plane and did not have physical significance. It is clear that a mapping satisfying the two formulated requirements does not exist. To maintain two parallel beams it is necessary to impart to the plane a charge of surface density parallel to and equidistant from their boundaries.

$$q = -\frac{1}{2\pi}\frac{\partial\Phi}{\partial y}\bigg|_{y=h} \qquad (\lim_{x\to\infty} q(x) = 0)$$

(2h is the distance between beams).

The problem of coupling the elements of a periodic focusing system is solved in exactly the same way. Reference [20] gives an approximate solution of the problem of periodic focusing. This solution is based on the approximation of the potential at the beam boundary by an expression, which in terms of the dimensionless variables used in § 1, takes the form

$$\varphi = 1 - (1 - \alpha)\cos(\pi x / 2\sigma). \tag{2.1}$$

The corresponding approximate solution for the Laplace equation is

$$\Phi(x, y) = 1 - (1 - \alpha)\cos(\pi x / 2\sigma)\,\mathrm{ch}(\pi y / 2\sigma). \tag{2.2}$$

The error involved in using expression (2.1) instead of the exact solution (1.1) for $\alpha = 0.25$ does not exceed 3%. Representing the potential by (2.1) ensures that the solution is periodic. The curves $\Phi(x, y) = \text{const}$, determined by (2.2) are represented by dashes in Figs. 3 and 6 for $\alpha = 0.2$ and $\alpha = 0.8$. It is clear that the electrodes with potential $\Phi = 1$ are the planes $x = 2\sigma$.

In accordance with what has been said above, in the exact solution the plane $x - 2\sigma$ must be a charged and not an equipotential surface in order to have an exact solution. The law of variation of the potential on this plane is then given by the expressions

$$\begin{aligned}\sigma = 2^{-1/2}\Big[&\Big(\sqrt{\tfrac{1}{2}(r+\Phi)} + 2\alpha^{1/2}\Big)\sqrt{R + \sqrt{\tfrac{1}{2}(r+\Phi)} - \alpha^{1/2}} - \\ &- \sqrt{\tfrac{1}{2}(r-\Phi)}\sqrt{R - \sqrt{\tfrac{1}{2}(r+\Phi)} + \alpha^{1/2}}\Big]\\ y = 2^{-1/2}\Big[&\Big(\sqrt{\tfrac{1}{2}(r+\Phi)} + 2\alpha^{1/2}\Big)\sqrt{R - \sqrt{\tfrac{1}{2}(r+\Phi)} + \alpha^{1/2}} + \\ &+ \sqrt{\tfrac{1}{2}(r-\Phi)}\sqrt{R + \sqrt{\tfrac{1}{2}(r+\Phi)} - \alpha^{1/2}}\Big].\end{aligned} \tag{2.3}$$

Curves $\Phi(y)$ given by (2.3) are shown in Fig. 8 for different values of α. The surface charge density in the plane $x = 2\sigma$ is

$$q = -\frac{1}{2\pi}\frac{\partial\Phi}{\partial x}\bigg|_{x=2\sigma} \qquad (\lim_{y\to\infty} q(y) = 0),$$

where Φ is given by expressions (1.2).

The results represented in Figs. 3 and 6 make it possible to compare the exact and approximate solutions. It is clear that the difference between them increases as x approaches unity. Here, the periodicity of the approximate solution artifically introduced in [20] is operative. Figure 9 represents one possible method of periodic focusing. The surface $\Phi = \text{const} < \alpha$ are used as low-voltage electrodes. The planes $x = 2k\sigma$ with variable potential, which shield one electrode from another, may take the form of a dense grid, the potential on which varies in accordance with expression (2.3).

The approximate solution of [18] can also be used to construct a periodic focusing system, as in [21]. In this case the high-voltage electrode takes the form of a biconvex lens, which is not to be recommended owing to its thickness. Clearly, such electrodes will introduce considerable perturbation not only at a distance from the beam but also at its boundary. Evidently, the exact solution for periodic focusing of a cylindrical beam, which can be constructed on the basis of the results of [5, 7], will also differ considerably from the approximate solution of [20]. Here, as above, the planes $x = 2k\sigma$ will not be equipotential surfaces.

REFERENCES

1. J. R. Pierce, "Rectilinear electron flow in beams," J. Appl. Phys., vol. 11, 1940.
2. R. J. Lomax, "Exact electrode systems for the formation of a curved space-charge beam," J. Electr. Contr., vol. 3, no. 4, 1957.
3. R. J. Lomax, "Exact electrode systems for the formation of a curved space-charge beam," J. Electr. Contr., vol. 7, no. 6, 1959.
4. P. T. Kirstein, "On the determination of the electrodes required to produce a given electric field distribution along a prescribed curve," Proc. IRE, vol. 46, 1958.
5. D. E. Radley, "The theory of the Pierce-type electron gun," J. Electr. Contr., vol. 4, no. 2, 1958.
6. D. E. Radley, "Electrodes for convergent Pierce-type guns," J. Electr. Contr., vol. 15, no. 5, 1963.
7. K. J. Harker, "Solution of the Cauchy problem for Laplace's equation in axially symmetric systems," J. Math. Phys., vol. 4, no. 7, 1963.
8. G. S. Kino, "A design method for crossed-field electron guns," IRE Trans. Electr. Dev., vol. ED-7, no. 3, 1960.
9. G. S. Kino and N. J. Taylor, "The design and performance of a magnetron injection gun," IRE Trans. Electr. Dev., vol. ED-9, no. 1, 1962.
10. J. Rosenblatt, "Three-dimensional space-charge flow," J. Appl. Phys., vol. 31, no. 8, 1960.
11. G. Kent, "Generalized Brillouin flow," Communication and Electronics, vol. 79, no. 48, 1960.
12. K. Pöschl and W. Veith, "Generalized Brillouin flow," J. Appl. Phys., vol. 33, no. 3, 1962.
13. E. Sugata, M. Terada, H. Hamade, and H. Pack, "Analytical design of electron gun with temperature-limited cathode," Technology Rep. Osaka Univ., vol. 12, no. 495, 1962.
14. V. A. Syrovoi, "Invariant-group solutions of the equations of a plane stationary beam of charged particles," PMTF, no. 4, 1962.
15. V. A. Syrovoi, "Solution of the Pierce problem for a ribbon beam under arbitrary emission conditions," PMTF, no. 2, 1964.
16. V. A. Syrovoi, "Invariant-group solutions of the equations of a nonstationary beam of charged particles," PMTF, no. 1, 1964.
17. C. E. Fay, A. L. Samuel and W. Shockley, "On the theory of space-charge between parallel plane electrodes," Bell System Techn. J., vol. 17, no. 1, 1938.
18. S. I. Molokovskii, "Analyticity in calculating electrode geometry for electrostatically focused ribbon beams," Radiotekhn. i elektron., vol. 7, no. 6, 1962.

19. W. W. Siekanowicz and F. E. Vaccaro, "Periodic electrostatic focusing of laminar parallel-flow electron beams," Proc. IRE, vol. 47, no. 3, 1959.

20. W. W. Siekanowicz, "Derivation of ideal electrode shapes for electrostatic beam focusing," RCA Rev., vol. 23, no. 1, 1962.

21. S. I. Molokovsky, "Electrostatic focusing systems for intense electron beams," J. Instn. Telecommunic. Engrs., vol. 9, no. 4, 1963.

22. V. P. Ignatenko, "Formation of a relativistic ribbon beam of charged particles," Radiotekhn. i elektron., vol. 8, no. 1, 1963.

23. J. Hadamard, Lectures on Cauchy's Problem in Linear Partial Differential Equations, Dover Publication, New York, 1952.

24. I. G. Petrovskii, Lectures on Equations in Partial Derivatives [in Russian], Fizmatgiz, 1961.

25. F. Berz, "Note on the potential derived from axial values in electron optics," Philos. Mag., vol. 41, no. 314, 1950.

26. R. Hechtel, "Zur Bestimmung der Elektrodenformen von Elektronenkanonen nach Pierce," Telefunken Zeitung, B 28, H 110, 1955.

27. G. R. Brewer, "Note on the determination of electrode shapes for a Pierce-type electron gun," J. Appl. Phys., vol. 28, no. 5, 1957.

28. Ho Kuo-Chu and R. J. Moon, "Electrostatic potential plotting for use in electron optical systems," J. Appl. Phys., vol. 24, no. 9, 1953.

29. B. Meltzer, "The stability of computation of the Pierce-Cauchy problem," J. Electr. Contr., vol. 8, no. 6, 1960.

30. B. Meltzer and A. R. Dinnis, "A note on Pierce electrode design," J. Electr. Contr., vol. 4, no. 5, 1958.

31. K. J. Harker, "Determination of electrode shapes for axially symmetric electron guns," J. Appl. Phys., vol. 31, no. 12, 1960.

32. K. J. Harker, "Electrode design for axially symmetric electron guns," J. Appl. Phys., vol. 33, no. 5, 1962.

33. E. R. Harrison, "Approximate electrode shapes for cylindrical electron beam," Brit. J. Appl. Phys., vol. 5, no. 1, 1954.

34. P. N. Daykin, "Electrode shapes for cylindrical electron beam," Brit. J. Appl. Phys., vol. 6, no. 7, 1955.

35. I. Sugai, "Numerical analysis for design of electron guns with curved electron trajectories," Proc. IRE, vol. 47, no. 1, 1959.

36. J. R. Hechtel, "Electrostatic focusing of microwave tubes," Microwave J., vol. 3, no. 12, 1960.

37. V. P. Ignatenko, "Obtaining convergent flat beams of charged particles with high current density," Zh. tekhn. fiz., vol. 31, no. 1, 1962.

38. G. Zepp, "Production de faisceaux électroniques denses en régime relativiste," C. R. Acad. Sci., tome 256, no. 16, 1963.

39. P. T. Kirstein, "Comment on a paper of D. E. Radley, 'The Theory of the Pierce Type Electron Gun,'" J. Electr. Contr., vol. 5, no. 2, 1958.

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